

Frame formalism for the N-dimensional quantum Euclidean spaces

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Abstract

We sketch our application [1] of a non-commutative version of the Cartan ‘moving-frame’ formalism to the quantum Euclidean space \mathbb{R}_q^N , the space which is covariant under the action of the quantum group $SO_q(N)$. For each of the two covariant differential calculi over \mathbb{R}_q^N based on the R -matrix formalism, we summarize our construction of a frame, the dual inner derivations, a metric and two torsion-free almost metric compatible covariant derivatives with a vanishing curvature. To obtain these results we have developed a technique which fully exploits the quantum group covariance of \mathbb{R}_q^N . We first find a frame in the larger algebra $\Omega^*(\mathbb{R}_q^N) \rtimes U_q so(N)$. Then we define homomorphisms from $\mathbb{R}_q^N \rtimes U_q^\pm so(N)$ to \mathbb{R}_q^N which we use to project this frame in $\Omega^*(\mathbb{R}_q^N)$.

1 Introduction and preliminaries

Non-commutative geometry as a way to describe the structure of space-time at small distances has been proposed since 1947 [2]. The original claim however made by him that it would serve as a Lorentz-invariant cut-off has been recently the object of some controversy. Here we briefly describe our application [1] of a non-commutative generalization [3] of the Cartan ‘moving-frame’ formalism to the quantum Euclidean spaces \mathbb{R}_q^N [4], the spaces which are covariant under the quantum group $SO_q(N)$.

In Section 1.1 we recall the basic concepts of non-commutative geometry, i.e. given a non-commutative algebra and a differential calculus on it how one can introduce the notions of a moving-frame or ‘Stehbein’ [3], of a corresponding metric and of a covariant derivative. For a more detailed exposition see [5]. In Section 1.2 we review the definition of the quantum Euclidean spaces [4] and of the differential calculi [6, 7, 8] on them. Finally in Section 2 we show how to construct the frame on \mathbb{R}_q^N and with its help the corresponding metric and covariant derivatives. It is necessary to enlarge the algebra with a ‘dilator’, the square roots and the inverses of some elements and in the case of even N also with one of the components of the angular momentum.

The quantum group covariance is an essential ingredient of our construction. We first define a frame in the cross-product $\Omega^1(\mathbb{R}_q^N) \rtimes U_q so(N)$ and then project it to a frame in $\Omega^1(\mathbb{R}_q^N)$ through homomorphisms φ^\pm from $\mathbb{R}_q^N \rtimes U_q^\pm so(N)$ to \mathbb{R}_q^N , which we apply to the components. In other words the components of the frame in the dx^i basis automatically provide a ‘local realization’ of $U_q^\pm so(N)$ in the extended algebra of \mathbb{R}_q^N , i.e. they satisfy the ‘RLL’ and the ‘gLL’ relations fulfilled by the Faddeev-Reshetikhin-Takhtadjan [4] generators of $U_q^\pm so(N)$. For odd N we have the interesting result that φ^\pm can be glued to a homomorphism $\varphi : \mathbb{R}_q^N \rtimes U_q so(N) \rightarrow \mathbb{R}_q^N$.

We recover for \mathbb{R}_q^N the formal ‘Dirac operator’ [9], as it had already been found in [11, 12]. In this way we construct a link between the approach to noncommutative geometry of Woronowicz [13], which is based on the quantum group covariance, and the one of Connes [9], which is based on the ‘Dirac operator’. Moreover, this method possibly suggests the correct choice of the physical coordinates [16]. For an application of this formalism to the development of a quantum field theory, see e.g. [17]. Note that the role of frame in the Woronowicz bicovariant differential calculi is played by the left- (or right-) invariant 1-forms; this has been applied to multiparametric deformations (including as a particular case the one-parameter one at hand) of the quantum Euclidean space in Ref. [14, 15].

1.1 The Cartan moving-frame formalism

Following [3], we briefly review here a non-commutative generalization of the moving-frame formalism of E. Cartan. The starting points are a non-

commutative algebra \mathcal{A} , which in the commutative limit should reduce to the algebra of functions on a parallelizable manifold M , and a differential calculus $\Omega^*(\mathcal{A})$ on it, which in the same limit should become the de Rham differential calculus on M . The set of 1-forms $\Omega^1(\mathcal{A})$ is assumed then to be a free \mathcal{A} -module of rank N . In addition we postulate the existence of a particular basis, a ‘frame’ or ‘Stehbein’, which we denote with $\{\theta^a\}_{1 \leq a \leq N}$ and which has the property of commuting with the elements of \mathcal{A} :

$$[f, \theta^a] = 0. \quad (1.1)$$

This basis θ^a is also required to be dual to a set of inner derivations $e_a = \text{ad } \lambda_a$:

$$df = e_a f \theta^a = [\lambda_a, f] \theta^a \quad \forall f \in \mathcal{A}. \quad (1.2)$$

The integer N can be interpreted as the dimension of M .

Under these assumptions a formal ‘Dirac operator’ [9] can be defined

$$\theta = -\lambda_a \theta^a. \quad (1.3)$$

It has the property

$$df = -[\theta, f]. \quad (1.4)$$

We shall require the center $\mathcal{Z}(\mathcal{A})$ of \mathcal{A} to be trivial: $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$. If necessary, one can enlarge the algebra until it is so.

Using the frame, the relations constraining the wedge product (we omit the symbol \wedge) will take the form

$$\theta^a \theta^b = P^{ab}_{cd} \theta^c \theta^d, \quad P^{ab}_{cd} \in \mathcal{Z}(\mathcal{A}). \quad (1.5)$$

The matrix P is necessarily a projector, and goes to the antisymmetric one in the commutative limit. We shall denote by $\pi : \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A})$ the \mathcal{A} -bilinear projector such that $\pi(\theta^a \otimes \theta^b) = \theta^a \theta^b$. Then for the condition $d^2 = 0$ to hold the λ_a have to satisfy a quadratic relation of the form

$$2\lambda_c \lambda_d P^{cd}_{ab} - \lambda_c F^c_{ab} - K_{ab} = 0, \quad F^c_{ab}, K_{ab} \in \mathcal{Z}(\mathcal{A}). \quad (1.6)$$

For the quantum Euclidean spaces \mathbb{R}_q^N the linear and constant terms will vanish:

$$F^c_{ab} = K_{ab} = 0. \quad (1.7)$$

The metric is defined as a non-degenerate \mathcal{A} -bilinear map

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \mathcal{A}. \quad (1.8)$$

It is completely determined if we assign its value in a basis

$$g(\theta^a \otimes \theta^b) =: g^{ab}. \quad (1.9)$$

As a further step, one has to introduce a ‘deformed flip’, i.e. an \mathcal{A} -bilinear map [10]

$$\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}), \quad \sigma(\theta^a \otimes \theta^b) =: S^{ab}_{cd} \theta^c \otimes \theta^d \quad (1.10)$$

which we suppose [18] to satisfy the braid relation. Bilinearity implies that $g^{ab} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$ and $S^{ab}_{cd} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$. Notice that in the classical limit the bilinearity condition, which e.g. for the metric reads,

$$g(f\xi \otimes \eta h) = f g(\xi \otimes \eta) h, \quad (1.11)$$

amounts to locality

$$[g(f\xi \otimes \eta h)](x) = f(x) [g(\xi \otimes \eta)](x) h(x),$$

what can be seen as a rationale for the definition given above. The flip is necessary in order to construct a covariant derivative D [19], i.e.. a map

$$D : \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \quad (1.12)$$

satisfying a left and right Leibniz rule:

$$D(f\xi) = df \otimes \xi + f D\xi, \quad D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f. \quad (1.13)$$

Then a consistent torsion map can be found

$$\Theta : \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A}), \quad \Theta = d - \pi \circ D. \quad (1.14)$$

where bilinearity requires that

$$\pi \circ (\sigma + 1) = 0. \quad (1.15)$$

The associated curvature map is then given by

$$\text{Curv} \equiv D^2 = \pi_{12} \circ D_2 \circ D, \quad \text{Curv}(\theta^a) = -\frac{1}{2} R^a_{bcd} \theta^c \theta^d \otimes \theta^b. \quad (1.16)$$

Here D_2 is a continuation of the map (1.10) to the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$

$$D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta). \quad (1.17)$$

In the case (1.7), a torsion-free covariant derivative [19] is

$$D\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta). \quad (1.18)$$

The covariant derivative is compatible with the metric if $g_{23} \circ D_2 = d \circ g$. Under the assumption (1.7) this is equivalent to [20]

$$S^{ae}_{df} g^{fg} S^{bc}_{eg} = g^{ab} \delta^c_d. \quad (1.19)$$

1.2 The quantum Euclidean spaces

Now, following [4], we briefly review the definition and some of the main results about the N -dimensional quantum Euclidean space \mathbb{R}_q^N .

The first building block is the matrix \hat{R} [4] for $SO_q(N, \mathbb{C})$ and its projector decomposition:

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t. \quad (1.20)$$

with \mathcal{P}_s , \mathcal{P}_a , \mathcal{P}_t $SO_q(N)$ -covariant q -deformations of the symmetric trace-free, antisymmetric and trace projectors respectively. The matrix \mathcal{P}_t can be expressed using the metric matrix g_{ij} . Explicitly the latter reads

$$g_{ij} = g^{ij} = q^{-\rho_i} \delta_{i,-j}. \quad (1.21)$$

It is a $SO_q(N)$ -isotropic tensor and is a deformation of the ordinary Euclidean metric. Here and in the sequel n is the rank of $SO(N, \mathbb{C})$, the indices take the values $i = -n, \dots, -1, 0, 1, \dots, n$ for N odd, and $i = -n, \dots, -1, 1, \dots, n$ for N even. Moreover, we have introduced the notation $(\rho_i) = (n - \frac{1}{2}, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, -\frac{1}{2} - n)$ for N odd, $(n - 1, \dots, 0, 0, \dots, 1 - n)$ for N even.

The N -dimensional quantum Euclidean space is defined as the associative algebra \mathbb{R}_q^N generated by elements $\{x^i\}_{i=-n, \dots, n}$ with relations

$$\mathcal{P}_{a_{kl}}^{ij} x^k x^l = 0. \quad (1.22)$$

This choice for the values of the indices is the most natural, because it is directly related to the weight vector associated to the Cartan subalgebra of the quantum group.

There are [6] two $SO_q(N)$ -covariant quadratic differential calculi on \mathbb{R}_q^N : $\Omega^*(\mathbb{R}_q^N)$ and $\bar{\Omega}^*(\mathbb{R}_q^N)$. They are generated by $\xi^i = dx^i$ and $\bar{\xi}^i = d\bar{x}^i$ respectively, with relations

$$x^i \xi^j = q \hat{R}_{kl}^{ij} \xi^k x^l, \quad \mathcal{P}_{a_{kl}}^{ij} \xi^k \xi^l = \xi^i \xi^j \quad \text{for } \Omega^1(\mathbb{R}_q^N), \quad (1.23)$$

$$x^i \bar{\xi}^j = q^{-1} \hat{R}^{-1 ij}_{kl} \bar{\xi}^k x^l, \quad \mathcal{P}_{a_{kl}}^{ij} \bar{\xi}^k \bar{\xi}^l = \xi^i \xi^j \quad \text{for } \bar{\Omega}^1(\mathbb{R}_q^N). \quad (1.24)$$

The nilpotency of d then automatically fixes the wedge product relations respectively among the ξ^i and the $\bar{\xi}^i$.

For $q \in \mathbb{R}^+$ \mathbb{R}_q^N can be equipped with a consistent conjugation with the help of the metric to lower and raise the indices $(x^i)^* = x^j g_{ji}$. In this way we obtain what is known as real quantum Euclidean space. The $*$ -structure can be extended to $\Omega^1(\mathbb{R}_q^N) \oplus \bar{\Omega}^1(\mathbb{R}_q^N)$ by setting $(\xi^i)^* = \bar{\xi}^j g_{ji}$. Then the two calculi are conjugate.

2 Application of the formalism to \mathbb{R}_q^N

In this section we shall actually construct a frame θ^a and the dual inner derivations $e_a = \text{ad } \lambda_a$ satisfying the conditions in Section 1.2 for \mathbb{R}_q^N .

However, before this is possible, we have to enlarge the algebra of \mathbb{R}_q^N . In Section 1.2 we required the center of the algebra \mathcal{A} to be trivial. But the algebra generated by the x^i has a nontrivial center, the element $r^2 = g_{kl}x^kx^l$ commutes with all coordinates. For this reason the formalism cannot be directly applied: with a general Ansatz of the type

$$\theta^a = \theta_i^a \xi^i \quad (2.25)$$

it is an immediate result that the condition (1.1) cannot be fulfilled for r^2 . To solve this problem we add to the algebra an element Λ , that we call the “dilatator”, and its inverse Λ^{-1} , satisfying the commutation relations

$$\begin{aligned} x^i \Lambda &= q \Lambda x^i, \\ \xi^i \Lambda &= \Lambda \xi^i, \quad \Lambda d = q d \Lambda. \end{aligned} \quad (2.26)$$

Note that Λ does not satisfy the Leibniz rule $d(fg) = f dg + (df)g \forall f, g \in \mathbb{R}_q^N$, so it must be interpreted as an element of the q -deformed Heisenberg algebra; in fact Λ^{-2} can be constructed [7] as a simple polynomial in the $U_q so(N)$ -covariant coordinates and partial derivatives. The choice in the second line of (2.26) is not unique. Another possibility [21], which was considered also in [15], would be $d\Lambda = \Lambda d$ and $\xi^i \Lambda = q \Lambda \xi^i$, which gives a strict Leibniz rule also for $f = \Lambda$. However, this is a bit cumbersome because it implies $df = 0$ to hold not only for the analogues of the constant functions.

Now, we can proceed with the actual construction of a frame. We first find one in the larger algebra $\Omega^*(\mathbb{R}_q^N) \rtimes U_q so(N)$. It is convenient to introduce the Faddeev-Reshetikin-Takhtadjan generators [4] of $U_q so(N)$:

$$\mathcal{L}^{+a}_l := \mathcal{R}^{(1)} \rho_l^a(\mathcal{R}^{(2)}) \quad \mathcal{L}^{-a}_l := \rho_l^a(\mathcal{R}^{-1(1)}) \mathcal{R}^{-1(2)}, \quad (2.27)$$

where $U_q^+ so(N)$ and $U_q^- so(N)$ are the Borel subalgebras of $U_q so(N)$, $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in U_q^+ so(N) \otimes U_q^- so(N)$ is the universal \mathcal{R} -matrix for $SO_q(N)$ and ρ the vector N -dimensional representation of $U_q so(N)$. Then

Proposition 1 [1] *If we define*

$$\vartheta^a = \Lambda^{-1} \mathcal{L}^{-a}_l \xi^l \in \Omega^1(\mathbb{R}_q^N) \rtimes U_q^- so(N) \quad (2.28)$$

$$\bar{\vartheta}^a = \Lambda \mathcal{L}^{+a}_l \bar{\xi}^l \in \bar{\Omega}^1(\mathbb{R}_q^N) \rtimes U_q^+ so(N), \quad (2.29)$$

then $\vartheta^a, \bar{\vartheta}^a$ constitute (in a generalized sense) a frame for \mathbb{R}_q^N :

$$[\vartheta^a, f] = 0, \quad [\bar{\vartheta}^a, f] = 0 \quad \forall f \in \mathbb{R}_q^N. \quad (2.30)$$

Moreover,

$$[\vartheta^a, \Lambda] = 0, \quad [\bar{\vartheta}^a, \Lambda] = 0. \quad (2.31)$$

The commutation relations among the ϑ^a (resp. $\bar{\vartheta}^a$) are as the ones among the ξ^i (resp. $\bar{\xi}^i$), except for the opposite products:

$$\begin{aligned} \mathcal{P}_{s_{ab}}^{cd} \vartheta^b \vartheta^a &= 0, & \mathcal{P}_{t_{ab}}^{cd} \vartheta^b \vartheta^a &= 0 \\ \mathcal{P}_{s_{ab}}^{cd} \bar{\vartheta}^b \bar{\vartheta}^a &= 0, & \mathcal{P}_{t_{ab}}^{cd} \bar{\vartheta}^b \bar{\vartheta}^a &= 0. \end{aligned} \quad (2.32)$$

Now, we wish to find a frame in the smaller algebra $\Omega^1(\mathbb{R}_q^N)$. To this end, it would suffice to replace in the definitions (2.28-2.29) $\mathcal{L}_l^{\pm a}$ by elements in \mathbb{R}_q^N which have their same commutation relations with any element $f \in \mathbb{R}_q^N$. Such elements exist since in Ref. [1] we have found homomorphisms $\varphi^\pm : \mathbb{R}_q^N \rtimes U_q^\pm so(N) \rightarrow \mathbb{R}_q^N$ acting as the identity on \mathbb{R}_q^N , so we just need to replace $\mathcal{L}_l^{\pm a}$ by $\varphi^\pm(\mathcal{L}_l^{\pm a})$ in order to project ϑ^a and $\bar{\vartheta}^a$ to frames in $\Omega^1(\mathbb{R}_q^N)$, $\bar{\Omega}^1(\mathbb{R}_q^N)$ respectively. Strictly speaking, the homomorphisms take values in a slightly enlarged version of the algebra of \mathbb{R}_q^N , obtained by adding some new elements $(r_i)^{\frac{1}{2}}$, $i = 0, \dots, n$, together with their inverses $(r_i)^{-\frac{1}{2}}$, fulfilling the condition that $r_i^2 = \sum_{k,l=-i}^i g_{kl} x^k x^l$. Their commutation relations with the x^j are automatically fixed by the latter condition and by the commutation relations between r_i^2 and x^j which can be drawn from (1.22). They read

$$\begin{aligned} x^j r_i &= r_i x^j \text{ for } |j| \leq i, \\ x^j r_i &= q r_i x^j \text{ for } j < -i, \\ x^j r_i &= q^{-1} r_i x^j \text{ for } j > i. \end{aligned} \quad (2.33)$$

But now for N even the center of the algebra is no longer trivial, even after the addition of Λ , because $r_1^{-1} x^{\pm 1}$ commutes with Λ as well as with the coordinates. In this case it is necessary to enlarge the algebra again, and we choose to add a Drinfeld-Jimbo generator $K = q^{\frac{H_1}{2}}$ and its inverse K^{-1} , where H_1 belongs to the Cartan subalgebra of $U_q so(N)$ and represents the component of the angular momentum in the $(-1, 1)$ -plane. This new element satisfies the commutation relations

$$K\Lambda = \Lambda K, \quad Kx^{\pm 1} = q^{\pm 1} x^{\pm 1} K, \quad Kx^{\pm i} = x^{\pm i} K \text{ for } i > 1 \quad (2.34)$$

and we fix its commutation relations with the 1-forms to be

$$K\xi^1 = q^{\pm 1} \xi^{\pm 1} K, \quad K\xi^i = \xi^i K \quad \text{for } i > 1. \quad (2.35)$$

Theorem 1 [1] *A homomorphism $\varphi^- : \mathbb{R}_q^N \rtimes U_q^- so(N) \rightarrow \mathbb{R}_q^N$ can be defined by setting on the generators*

$$\varphi^-(a) = a, \quad \forall a \in \mathbb{R}_q^N, \quad (2.36)$$

$$\varphi^-(\mathcal{L}_j^{-i}) = g^{ih} \Lambda^{-1} [\lambda_h, x^k] g_{kj}, \quad (2.37)$$

with

$$\begin{aligned} \lambda_0 &= \gamma_0 \Lambda (x^0)^{-1} && \text{for } N \text{ odd,} \\ \lambda_{\pm 1} &= \gamma_{\pm 1} \Lambda (x^{\pm 1})^{-1} K^{\mp 1} && \text{for } N \text{ even,} \\ \lambda_a &= \gamma_a \Lambda r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} && \text{otherwise,} \end{aligned} \quad (2.38)$$

and $\gamma_a \in \mathbb{C}$ normalization constants fulfilling the conditions

$$\begin{aligned} \gamma_0 &= -q^{-\frac{1}{2}}h^{-1} && \text{for } N \text{ odd,} \\ \gamma_1\gamma_{-1} &= \begin{cases} -q^{-1}h^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} \\ \gamma_a\gamma_{-a} &= -q^{-1}k^{-2}\omega_a\omega_{a-1} && \text{for } a > 1. \end{aligned} \quad (2.39)$$

Here and in the sequel we set

$$h := \sqrt{q} - 1/\sqrt{q}, \quad k := q - q^{-1}, \quad \omega_i := q^{\rho_i} + q^{-\rho_i}. \quad (2.40)$$

The relation (2.39) fixes only the product $\gamma_a\gamma_{-a}$, γ_0^2 for N odd and $\gamma_1\gamma_{-1}$ for N even are positive real numbers, while all the remaining products $\gamma_a\gamma_{-a}$ are negative.

Theorem 2 [1] *A homomorphism $\varphi^+ : \mathbb{R}_q^N \rtimes U_q^+ so(N) \rightarrow \mathbb{R}_q^N$ by setting on the generators*

$$\varphi^+(a) = a, \quad \forall a \in \mathbb{R}_q^N, \quad (2.41)$$

$$\varphi^+(\mathcal{L}^{+i}_j) = g^{ih}\Lambda[\bar{\lambda}_h, x^k]g_{kj}, \quad (2.42)$$

with

$$\begin{aligned} \bar{\lambda}_0 &= \bar{\gamma}_0\Lambda^{-1}(x^0)^{-1} && \text{for } N \text{ odd,} \\ \bar{\lambda}_{\pm 1} &= \bar{\gamma}_{\pm 1}\Lambda^{-1}(x^{\pm 1})^{-1}K^{\pm 1} && \text{for } N \text{ even,} \\ \bar{\lambda}_a &= \bar{\gamma}_a\Lambda^{-1}r_{|a|}^{-1}r_{|a|-1}^{-1}x^{-a} && \text{otherwise,} \end{aligned} \quad (2.43)$$

and $\bar{\gamma}_a \in \mathbb{C}$ normalization constants fulfilling the conditions

$$\begin{aligned} \bar{\gamma}_0 &= q^{\frac{1}{2}}h^{-1} && \text{for } N \text{ odd,} \\ \bar{\gamma}_1\bar{\gamma}_{-1} &= \begin{cases} -qh^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} \\ \bar{\gamma}_a\bar{\gamma}_{-a} &= -qk^{-2}\omega_a\omega_{a-1} && \text{for } a > 1. \end{aligned} \quad (2.44)$$

Under which circumstances can these homomorphisms be ‘glued’ together to a homomorphism of the whole of $U_q so(N)$? The answer is given by

Theorem 3 [1] *In the case of odd N we one can define a homomorphism $\varphi : U_q so(N) \rtimes \mathbb{R}_q^N \rightarrow \mathbb{R}_q^N$ by setting on the generators*

$$\varphi(a) = a \quad \forall a \in \mathbb{R}_q^N \quad (2.45)$$

$$\varphi(\mathcal{L}^{-i}_j) = g^{ih}\Lambda^{-1}[\lambda_h, x^k]g_{kj}, \quad (2.46)$$

$$\varphi(\mathcal{L}^{+i}_j) = g^{ih}\Lambda[\bar{\lambda}_h, x^k]g_{kj}, \quad (2.47)$$

with $\lambda_j, \bar{\lambda}_j$ defined as in (2.38), (2.43) and with coefficients given by

$$\begin{aligned}\gamma_0 &= -q^{-\frac{1}{2}}h^{-1} \\ \gamma_1^2 &= -q^{-2}h^{-2} \\ \gamma_a^2 &= -q^{-2}\omega_a\omega_{a-1}k^{-2} \quad \text{for } a > 1 \\ \gamma_a &= q\gamma_{-a} \quad \text{for } a \leq 1 \\ \bar{\gamma}_a &= -q\gamma_a.\end{aligned}\tag{2.48}$$

Notice that the $\gamma_a, \bar{\gamma}_a$ for $a \neq 0$ are imaginary and fixed only up to a sign. Therefore, the homomorphism φ does not preserve the star structure of the real section $U_qso(N, \mathbb{R})$ for $q \in \mathbb{R}^+$.

In the case of even N it is not possible to extend φ^\pm to a homomorphism φ from the whole of $U_qso(N)$, due to the necessity of adding $K^{\pm 1}$ to the algebra. In other words in this case there is one of the generators of the Cartan subalgebra of $U_qso(N)$ which cannot be realized in \mathbb{R}_q^N and this has a consequence that it is not possible to satisfy the would-be φ -image of the commutation relations between \mathcal{L}^+i_j and \mathcal{L}^-h_k , so that the homomorphisms cannot be extended to the whole $U_qso(N)$.

The fact that the homomorphism φ^\pm or φ exist can be interpreted as the existence of a local realization of $U_q^\pm so(N)$ or $U_qso(N)$.

Summing up, for the frames of the two calculi we explicitly find:

$$\theta^a = \theta_l^a \xi^l = \Lambda^{-2} g^{ab} [\lambda_b, x^j] g_{jl} \xi^l, \quad \bar{\theta}^a = \bar{\theta}_l^a \bar{\xi}^l = \Lambda^2 g^{ab} [\bar{\lambda}_b, x^j] g_{jl} \bar{\xi}^l. \tag{2.49}$$

The θ^a commute both with the coordinates and with Λ .

From the preceding theorems one derives the commutation relations

Theorem 4 [1]

$$\mathcal{P}_{acd}^{ab} \lambda_a \lambda_b = 0, \quad \mathcal{P}_{acd}^{ab} \bar{\lambda}_a \bar{\lambda}_b = 0 \tag{2.50}$$

$$\mathcal{P}_{acd}^{ab} \theta^c \theta^d = \theta^a \theta^b, \quad \mathcal{P}_{acd}^{ab} \bar{\theta}^c \bar{\theta}^d = \bar{\theta}^a \bar{\theta}^b. \tag{2.51}$$

Hence the matrix P of equations (1.5) and (1.6) is the q -deformed antisymmetric projector \mathcal{P}_a . The elements $\lambda_a, \bar{\lambda}_a$ satisfy the same commutation relation as the x^i , while the $\theta^a, \bar{\theta}^a$ those satisfied by the ξ^i .

It is a very interesting result that applying a condition like (1.1), which does not depend on the quantum group covariance of \mathbb{R}_q^N , to the quantum group covariant differential calculi, the components θ_i^a of the frame automatically carry so to say a ‘local realization’ of $U_qso(N)$.

The Dirac operator [9] of 1.3 is easily verified to be given by [12, 11].

$$\theta = \omega_n q^{\frac{N}{2}} k^{-1} r^{-2} g_{ij} x^i \xi^j, \quad \text{for } \Omega^1(\mathbb{R}_q^N), \tag{2.52}$$

$$\bar{\theta} = -\omega_n q^{-\frac{N}{2}} k^{-1} r^{-2} g_{ij} x^i \bar{\xi}^j \quad \text{for } \bar{\Omega}^1(\mathbb{R}_q^N). \tag{2.53}$$

Finally we summarize the results found in [1] for the metric and the covariant derivative for each of the two calculi $\Omega(\mathbb{R}_q^N)$ and $\Omega^*(\mathbb{R}_q^N)$. In the

$\theta^a, \bar{\theta}^a$ basis respectively the actions of g and σ are

$$\sigma(\theta^a \otimes \theta^b) = S^{ab}_{cd} \theta^c \otimes \theta^d, \quad g(\theta^a \otimes \theta^b) = g^{ab} \quad \text{for } \Omega^*(\mathbb{R}_q^N), \quad (2.54)$$

$$\sigma(\bar{\theta}^a \otimes \bar{\theta}^b) = \bar{S}^{ab}_{cd} \bar{\theta}^c \otimes \bar{\theta}^d, \quad g(\bar{\theta}^a \otimes \bar{\theta}^b) = g^{ab} \quad \text{for } \bar{\Omega}^*(\mathbb{R}_q^N). \quad (2.55)$$

while in the $\xi^i, \bar{\xi}^i$ basis we get

$$g(\xi^i \otimes \xi^j) = g^{ij} \Lambda^2, \quad \sigma(\xi^i \otimes \xi^j) = S^{ij}_{hk} \xi^h \otimes \xi^k \quad \text{for } \Omega^*(\mathbb{R}_q^N), \quad (2.56)$$

$$g(\bar{\xi}^i \otimes \bar{\xi}^j) = g^{ij} \Lambda^{-2}, \quad \sigma(\bar{\xi}^i \otimes \bar{\xi}^j) = \bar{S}^{ij}_{hk} \bar{\xi}^h \otimes \bar{\xi}^k. \quad \text{for } \bar{\Omega}^*(\mathbb{R}_q^N). \quad (2.57)$$

Unfortunately, it is not possible to satisfy simultaneously the metric compatibility condition (1.19) and the bilinearity condition for the torsion (1.15). The best we can do is to weaken the compatibility condition to a condition of proportionality. Then for each calculus we find the two solutions for σ :

$$S = q\hat{R}, \quad \bar{S} = (q\hat{R})^{-1} \quad \text{for } \Omega^*(\mathbb{R}_q^N), \quad (2.58)$$

$$\bar{S} = q\hat{R}, \quad S = (q\hat{R})^{-1} \quad \text{for } \bar{\Omega}^*(\mathbb{R}_q^N). \quad (2.59)$$

This implies that the covariant derivatives and metric are compatible only up to a conformal factor:

$$S^{ae}_{df} g^{fg} S^{cb}_{eg} = q^{\pm 2} g^{ac} \delta^b_d, \quad \bar{S}^{ae}_{df} g^{fg} \bar{S}^{cb}_{eg} = q^{\pm 2} g^{ac} \delta^b_d. \quad (2.60)$$

The linear curvatures associated to the covariant derivatives defined by (1.18) for each the calculi vanish, consistently with the fact that \mathbb{R}_q^N should be flat.

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